Discrete Math: Homework 3

Tuesday/Thursday 11:00-12:15, Phillips 383

Reese Lance - Section 003

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Unit 1.7

#8

Prove that if n is a perfect square, then $n + 2$ is not a perfect square.

Proof. Let n be a perfect square. Given P(x) = $\exists x \in \mathbb{Z}, n = x^2$ and Q(y) = $\exists y \in \mathbb{Z}, n + 2 = y^2$, we will prove that $\forall n \in \mathbb{Z}(P(n) \implies \neg Q(n))$ using a proof by contradiction.

Proof by Contradiction:

$$\begin{aligned} \text{Assume } \exists n \in \mathbb{Z}(P(n) \implies Q(n)). && \text{Premise.} \\ P(x) = \exists x \in \mathbb{Z}, n = x^2 && \text{Premise.} \\ Q(y) = \exists x \in \mathbb{Z}, n + 2 = y^2 && \text{Premise.} \\ x^2 + 2 = y^2 && \text{Algebraic Substitution.} \\ y^2 - x^2 = 2 && \text{Algebraic Equivalence.} \\ (y + x)(y - x) = 2 && \text{Factorization.} \end{aligned}$$

Since y and x are integers, $y + x$ and $y - x$ are integers. The only integer factors of 2 are 1 and 2. Therefore,

$$\begin{aligned} y + x = 1 \\ y - x = 2 \end{aligned}$$

Using algebraic equivalence, we can solve for y.

$$y = \frac{3}{2} \qquad \perp$$

Since y is not an integer, we have reached a contradiction. Therefore, $\forall n \in \mathbb{Z}(P(n) \implies \neg Q(n))$. □

#14
Prove that if x is rational and $x \neq 0$, then $1/x$ is rational.

Proof. Let x be rational and $x \neq 0$. Given $P(x) = \exists p, q \in \mathbb{Z}, p, q \neq 0, x = p/q$ and $Q(y) = \exists r, s \in \mathbb{Z}, r, s \neq 0, 1/x = r/s$,k we will prove that $\forall x \in \mathbb{Q}(P(x) \implies Q(x))$ using a direct proof.
Direct Proof:

$$
\begin{aligned}
x &= \frac{p}{q} && \text{Premise.} \\
\frac{1}{x} &= \frac{q}{p} && \text{Algebraic Equivalence.} \\
\frac{1}{x} &= \frac{r}{s} && \text{Let } r = q \text{ and } s = p. \\
s &\neq 0 && \text{Since } p \neq 0. \\
\exists r, s \in \mathbb{Z}, r, s &\neq 0, 1/x = r/s && \text{Premise.} \\
\forall x \in \mathbb{Q}(P(x) &\implies Q(x)) && \text{Conclusion.}
\end{aligned}
$$

□

#20

Prove that if n is an integer and $3n + 2$ is even, then n is even using

a) a proof by contraposition

Proof by Contraposition:

Proof. Given $P(n) = \exists x \in \mathbb{Z}, 2x = 3n + 2$ and $Q(n) = \exists y \in \mathbb{Z}, 2y = n$, we will prove that $\forall n \in \mathbb{Z}(P(n) \implies Q(n))$ using a proof by contraposition. In fact, we want to show that if n is odd, then $3n + 2$ is odd.

$$\begin{aligned} \text{Assume } \exists n \in \mathbb{Z}(\neg Q(n) \implies \neg P(n)). & \quad \text{Premise.} \\ \neg Q(n) = \exists y \in \mathbb{Z}, n = 2y + 1 & \quad \text{Premise.} \end{aligned}$$

Now we need to prove $\neg P(n)$.

$$\begin{aligned} 3n + 2 &= 3(2y + 1) + 2 && \text{Algebraic Substitution.} \\ &= 6y + 5 && \text{Algebraic Equivalence.} \\ &= 2(3y + 2) + 1 && \text{Algebraic Equivalence.} \end{aligned}$$

Let $\bar{k} = 3y + 2, \bar{k} \in \mathbb{Z}$.

$$= 2\bar{k} + 1 \qquad \text{Algebraic Equivalence.}$$

Since $\bar{k}$ is an integer, $2\bar{k} + 1$ is odd. Therefore, $3n + 2$ is odd and we have proved the contrapositive. □

b) a proof by contradiction

Proof by Contradiction:

Proof. Given $P(n) = \exists x \in \mathbb{Z}, 2x = 3n + 2$ and $Q(n) = \exists y \in \mathbb{Z}, 2y = n$, we will prove that $\forall n \in \mathbb{Z}(P(n) \implies Q(n))$ using a proof by contradiction. In fact, we have to show a contradiction in the statement, if $3n + 2$ is even then n is odd.

$$\begin{aligned} &\text{Assume } \exists n \in \mathbb{Z}(P(n) \implies \neg Q(n)). && \text{Premise.} \\ &P(n) = \exists x \in \mathbb{Z}, 2x = 3n + 2 && \text{Premise.} \\ &\neg Q(n) = \exists y \in \mathbb{Z}, n = 2y + 1 && \text{Premise.} \end{aligned}$$

Now we need to find the contradiction in the statement.

$$\begin{aligned} 2x &= 3(2y + 1) + 2 && \text{Algebraic Substitution.} \\ &= 6y + 5 && \text{Algebraic Equivalence.} \\ &= 2(3y + 2) + 1 && \text{Algebraic Equivalence.} \end{aligned}$$

Let $\bar{k} = 3y + 2, \bar{k} \in \mathbb{Z}$.

$$2x = 2\bar{k} + 1 \qquad \bot$$

Here we have found the contradiction in the previous statement, since $2x$ is even and $2\bar{k} + 1$ is odd. Therefore, we have proved the proposition.

□

#30
Prove that $m^2 = n^2$ if and only if $m = n$ or $m = -n$.

Proof. Given $P(m,n) =$ "$m^2 = n^2$" and $Q(m,n) =$ "$m = n$" $\vee$ "$m = -n$", we will prove that $\forall m, n \in \mathbb{Z}(P(m,n) \iff Q(m,n))$ using a direct proof.
Direct Proof:
In the case of $P(m,n) \implies Q(m,n)$,

$$\begin{aligned} m^2 &= n^2 \\ m^2 - n^2 &= 0 \\ (m-n)(m+n) &= 0 \end{aligned}$$

Which yields,

$$\begin{aligned} m - n &= 0 \\ m &= n \\ &\textbf{or} \\ m + n &= 0 \\ m &= -n \end{aligned}$$

Therefore, $P(m,n) \implies Q(m,n)$.

In the case of $Q(m,n) \implies P(m,n)$ we can assume that $m = n$ without the loss of generality,

$$\begin{aligned} m &= n \\ m - n &= 0 \\ (m-n)(m+n) &= 0 \\ m^2 - n^2 &= 0 \\ m^2 &= n^2 \end{aligned}$$

Therefore, $Q(m,n) \implies P(m,n)$. Since $P(m,n) \implies Q(m,n)$ and $Q(m,n) \implies P(m,n)$, we have proved that $\forall m, n \in \mathbb{Z}(P(m,n) \iff Q(m,n))$. □

Unit 1.8

#8

Prove using the notion of without loss of generality that $5x + 5y$ is an odd integer when x and y are integers of opposite parity.

Proof. Given $P(x, y) = \exists k, l \in \mathbb{Z}, x = 2k, y = 2l + 1$ and $Q(x, y) = \exists w \in \mathbb{Z}, 2w + 1 = 5x + 5y$, we will prove that $\forall x, y \in \mathbb{Z}(P(x, y) \implies Q(x, y)$ is odd) using a direct proof. Without loss of generality, we can assume that x is even and y is odd.

Direct proof:

Assume $x = 2k, y = 2l + 1, k, l \in \mathbb{Z}$.	Premise.
$5x + 5y = 5(2k) + 5(2l + 1)$	Algebraic Substitution.
$= 10k + 10l + 5$	Algebraic Equivalence.
$= 2(5k + 5l + 2) + 1$	Algebraic Equivalence.

Let $\bar{k} = 5k + 5l + 2, \bar{k} \in \mathbb{Z}$.

$= 2\bar{k} + 1$	Algebraic Equivalence.

Therefore $5x + 5y$ is an odd integer when x and y are integers of opposite parity. □

#10

Prove that there is a positive integer that equals the sum of the positive integers not exceeding it. Is your proof constructive or nonconstructive?

Proof. Given $P(n) = \exists x \in \mathbb{Z}, x = \sum_{i=1}^{n} i$ and $Q(n) = \exists y \in \mathbb{Z}, y > 0$, we will prove that $\forall n \in \mathbb{Z}(P(n) \implies Q(n))$ using a direct proof.

Constructive Proof:

$$P(1) = \exists x \in \mathbb{Z}, x = \sum_{i=1}^{1} i \qquad \text{Given P(1).}$$

$$= 1$$

Using a **constructive** proof, we have shown that there exists a positive integer, 1, that equals the sum of the positive integers not exceeding it. ☐

#32
Prove that there are no solutions in integers x and y to the equation $2x^2 + 5y^2 = 14$.

Proof. Given $P(x, y) = \exists x, y \in \mathbb{Z}, 2x^2 + 5y^2 = 14$, we will prove that $\forall x, y \in \mathbb{Z}(\neg P(x, y))$ using a proof by exhaustion.
Proof by Exhaustion:
Given $2x^2 + 5y^2 = 14$,

$$|2x^2| \leq 14$$
$$|5y^2| \leq 14$$

Our possible values for x are $\{0, 1, 2\}$ and our possible values for y are $\{0, 1\}$.

$2(0)^2 + 5(0)^2 = 0 + 0 = 0$	$\neq 14$	$(0, 0)$
$2(0)^2 + 5(1)^2 = 0 + 5 = 5$	$\neq 14$	$(0, 1)$
$2(1)^2 + 5(0)^2 = 2 + 0 = 2$	$\neq 14$	$(1, 0)$
$2(1)^2 + 5(1)^2 = 2 + 5 = 7$	$\neq 14$	$(1, 1)$
$2(2)^2 + 5(0)^2 = 8 + 0 = 8$	$\neq 14$	$(2, 0)$
$2(2)^2 + 5(1)^2 = 8 + 5 = 13$	$\neq 14$	$(2, 1)$

Therefore, $\forall x, y \in \mathbb{Z}(\neg P(x, y))$. □

#36
Prove that $\sqrt[3]{2}$ is irrational.

Proof. Given P(x) $= x \notin \mathbb{Q}$, we can do a proof by contradiction to prove that $\sqrt[3]{2}$ is irrational. Given $\neg P(x) = \exists p, q \in \mathbb{Z}, q \neq 0, x = \frac{p}{q}$
Proof by Contradiction:

$\sqrt[3]{2} = \dfrac{p}{q}$	Given $\neg P(x)$
$2 = \dfrac{p^3}{q^3}$	Cubing both sides.
$2q^3 = p^3$	Multiplying both sides by q^3.
p^3 is even.	Since $2\bar{k}$ is even.
p is even.	Since p^3 is even.

Given that p is even, Let $\bar{k} \in \mathbb{Z}, p = 2\bar{k}$.

$2q^3 = (2\bar{k})^3$	Algebraic Substitution.
$2q^3 = 8\bar{k}^3$	Algebraic Equivalence.
$q^3 = 4\bar{k}^3$	Algebraic Equivalence.

Since p is even, we have two cases where q is even and q is odd.

1. q is even.

$2\|p, 2\|q$	Since p and q are even.	
$\gcd(p, q) \geq 2$	Since p and q are even.	$\perp$

   We find the contradiction in the statement, since $\gcd(p, q) \geq 2$ and p/q is in lowest terms.

2. q is odd.

$\exists l \in \mathbb{Z}, q = 2l + 1$	Since q is odd.	
$(2l + 1)^3 = 4\bar{k}^3$	Algebraic Substitution.	
$8l^3 + 12l^2 + 6l + 1 = 4\bar{k}^3$	Algebraic Expansion.	
$8l^3 + 12l^2 + 6l - 4\bar{k}^3 = -1$	Algebraic Equivalence.	
$2(4l^3 + 6l^2 + 3l - 2\bar{k}^3) = -1$	Algebraic Equivalence.	
$\bar{l} \in \mathbb{Z}, \bar{l} = 4l^3 + 6l^2 + 3l - 2\bar{k}^3$	Define $\bar{l}$	
$2\bar{l} = -1$	Algebraic Equivalence.	$\perp$

   We find the contradiction in the statement, since $2\bar{l}$ cannot equal -1.

Therefore, we have proved that $\sqrt[3]{2}$ is irrational by contradiction. □