Discrete Math: Homework 5

Due on October 12, 2023 at 11:59pm

Tuesday/Thursday 11:00-12:15, Phillips 383

Reese Lance - Section 003

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Unit 2.2

#2 Suppose that A is the set of sophomores at your school and B is the set of students in discrete mathematics at your school. Express each of these sets in terms of A and B.

a) the set of sophomores taking discrete mathematics in your school

$$A \cap B$$

b) the set of sophomores at your school who are not taking discrete mathematics

$$A - B \text{ or } A \cap B^c$$

c) the set of students at your school who either are sophomores or are taking discrete mathematics

$$A \cup B$$

d) the set of students at your school who either are not sophomores or are not taking discrete mathematics

$$(A \cap B)^c$$

#14 Find the sets A and B if $A - B = \{1, 5, 7, 8\}$, $B - A = \{2, 10\}$, and $A \cap B = \{3, 6, 9\}$.

Given $A \cap B = \{3, 6, 9\}$, we also know that $A \cap B^c = \{1, 5, 7, 8\}$.

$$\begin{aligned}(A \cap B) \cup (A \cap B^c) &= A \cap (B \cup B^c) \\ &= A \cap U \\ &= A\end{aligned}$$

Given this, we can conclude that $A = \{1, 3, 5, 6, 7, 8, 9\}$. Given A, we can conclude that $B = \{2, 3, 6, 9, 10\}$.

#20 Let A, B and C be sets. Show that,

a) $(A \cup B) \subseteq (A \cup B \cup C)$

$$\begin{aligned}(A \cup B) \implies & \forall x, x \in A \vee x \in B \\ =& \forall x, x \in A \vee x \in B \vee x \in C \\ =& \forall x, x \in (A \cup B \cup C)\end{aligned}$$

$$\therefore (A \cup B) \subseteq (A \cup B \cup C)$$

d) $(A - C) \cap (C - B) = \emptyset$

$$\begin{aligned}(A - C) \cap (C - B) &\subset (A \cap C^c) \cap (C \cap B^c) \\ &\subset A \cap B \cap (C^c \cap C) \\ &\subset A \cap B \cap \emptyset \\ &\subset \emptyset\end{aligned}$$

The empty set is a subset of every set, so we can conclude that $(A - C) \cap (C - B) = \emptyset$.

e) $(B - A) \cup (C - A) = (B \cup C) - A$

First, $(B - A) \cup (C - A) \subset (B \cup C) - A$.

$$\begin{aligned}x \in (B - A) \cup (C - A) \\ x \in (B - A) \vee x \in (C - A) \\ x \in B \cap A^c \vee x \in C \cap A^c \\ (x \in B \wedge x \notin A) \vee (x \in C \wedge x \notin A) \\ (x \in B \vee x \in C) \wedge (x \notin A) \\ (x \in B \cup C) \wedge (x \notin A) \\ x \in (B \cup C) \cap A^c \\ x \in (B \cup C) - A \\ \therefore (B - A) \cup (C - A) \subset (B \cup C) - A\end{aligned}$$

Then, $(B - A) \cup (C - A) \supset (B \cup C) - A$.

$$\begin{aligned}x \in (B \cup C) - A \\ x \in (B \cup C) \wedge x \notin A \\ (x \in B \vee x \in C) \wedge (x \notin A) \\ (x \in B \wedge x \notin A) \vee (x \in C \wedge x \notin A) \\ x \in B \cap A^c \vee x \in C \cap A^c \\ x \in (B - A) \vee x \in (C - A) \\ x \in (B - A) \cup (C - A) \\ \therefore (B - A) \cup (C - A) \supset (B \cup C) - A\end{aligned}$$

Since we have proved either set is a subset of the other, we can conclude that $(B - A) \cup (C - A) = (B \cup C) - A$.

#54 Let $A_i = \{\ldots, -2, -1, 0, 1, \ldots, i\}$. Find

a) $\bigcup\limits_{i=1}^{n} A_i$

$$\begin{aligned}\bigcup_{i=1}^{n} A_i &= \{\ldots, -2, -1, 0, 1, \ldots, n\} \\ &= \{x \in \mathbb{Z} \mid x \leq n\} \\ &= A_n\end{aligned}$$

b) $\bigcap\limits_{i=1}^{n} A_i$

$$\begin{aligned}\bigcap_{i=1}^{n} A_i &= \{\ldots, -2, -1, 0, 1\} \\ &= \{x \in \mathbb{Z} \mid x \leq 1\} \\ &= A_1\end{aligned}$$

Unit 2.3

#2 Determine whether f is a function from $\mathbb{Z}$ to $\mathbb{R}$ if

a) $f(n) = \pm n$. Since a function cannot map to a single input, n, to two different outputs, n and $-n$, this is not a function.

b) $f(n) = \sqrt{n^2 + 1}$ This is a function, since for every input, n, there is only one output, $\sqrt{n^2 + 1}$, which is guaranteed to be a real number as for all integers, $n^2 + 1 \geq 1$.

c) $f(n) = \frac{1}{n^2 - 4}$ This is not a function from $\mathbb{Z}$ to $\mathbb{R}$, since $n^2 - 4 = 0$ when $n = \pm 2$, and thus $f(n)$ is undefined for $n = \pm 2$.

#6 Find the domain and range of each of these functions.

a) the function that assigns to each pair of positive integers the first integer of the pair

$$\text{Domain: } \mathbb{Z}^+ \times \mathbb{Z}^+$$
$$\text{Range: } \mathbb{Z}^+$$

d) the function that assigns to each positive integer the largest integer not exceeding the square root of the integer

$$\text{Domain: } \mathbb{Z}^+$$
$$\text{Range: } \mathbb{Z}^+$$

#10 Determine whether each of these functions from $\{a, b, c, d\}$ to itself is one-to-one.

a) $f(a) = b, f(b) = a, f(c) = c, f(d) = d$ Yes, every output has a unique input.

b) $f(a) = b, f(b) = b, f(c) = d, f(d) = c$ No, b can be produced by both a and b.

c) $f(a) = d, f(b) = b, f(c) = c, f(d) = d$ No, d can be produced by both a and d.

#16 Consider these functions from the set of students in a discrete mathematics class. Under what conditions is the function one-to-one if it assigns to a student his or her

a) **mobile phone number.** The function is one-to-one if no students share a phone number, which is expected.

b) **student identification number.** The function is one-to-one if no students share an ID number, which is expected.

c) **final grade in the class.** The function is one-to-one iff no students share a final grade in the class.

d) **home town.** The function is one-to-one iff no students share a home town.

#26

a) Prove that a strictly increasing function from $\mathbb{R}$ to itself is one-to-one.

Proof. Since the function is strictly increasing, we can assume that $\forall a, b \in \mathbb{R}, a > b \implies f(a) > f(b)$. We can prove that this function is one-to-one by proving that for all $x, y \in \mathbb{R}$, $f(x) = f(y) \implies x = y$. In order to use this, we'll use cases,

(a) $x > y$. Since f is strictly increasing, $f(x) > f(y)$, which contradicts $f(x) = f(y)$. Thus, $x > y$ is impossible.

(b) $x < y$. Since f is strictly increasing, $f(x) < f(y)$, which contradicts $f(x) = f(y)$. Thus, $x < y$ is impossible.

(c) $x = y$. This is trivially true.

Since $x > y$ and $x < y$ are impossible, we can conclude that given $f(x) = f(y)$, $x = y$, and thus f is one-to-one. □

b) Give an example of an increasing function from $\mathbb{R}$ to itself that is not one-to-one.

The piecewise function, $f(x) = \begin{cases} 0 & x \leq 0 \\ x & x > 0 \end{cases}$ is increasing, but not one-to-one as any two inputs less than or equal to 0 will produce the same output, 0.