Discrete Math: Notes

Tuesday/Thursday 11:00-12:15, Phillips 383

Reese Lance - Section 003

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Proposition 1. *The sum of the first n, positive odd integers is the equation n^2.*

Proof. By induction

1. base case

$$P(1) = 1^2 = 1$$

2. inductive step

$$\text{Assume } P(n) = \text{"}1 + 3 + 5 + ... + 2n - 1 = n^2\text{"}$$
$$\text{WTS: } P(n+1) = \text{"}1 + 3 + 5 + ... + 2n - 1 + 2(n+1) - 1 = (n+1)^2\text{"}$$
$$1 + 3 + 5 + ... + 2n - 1 + 2(n+1) - 1 = (n+1)^2$$

□

Proposition 2. *All horses are the same color.*
Note: *It suffices to show: P(n) = "All sets of n horses have the same color"*

Proof. By induction

1. base case:
   P(1) = "All sets of 1 horse have the same color"

2. inductive step: Assume P(n), i.e. every step of n horses have the same color
   WTS: P(n+1), i.e. every set of n+1 horses have the same color
   H = $\{H_1, H_2, ..., H_n, H_{n+1}\}$ is a set of n+1 horses
   $H_1 = \{H_1, H_2, ..., H_n\}$ is a set of n horses
   $H_2 = \{H_2, H_3, ..., H_n, H_{n+1}\}$ is a set of n horses

□

Theorem 1. *Given two sets, when n $\neq 1$, when they both overlap and are disjoint, the union of the two sets is equal to the sum of the two sets.*

Strong Induction

This is what we were doing before:
Weak Induction:

1. base case
2. inductive step ($P(n) \implies P(n+1)$)

Trying to prove $P(n) \forall n \in \mathbb{N}$.

If $P(n) \implies P(n+1)$ is too hard to show, instead try strong induction:

1. base case (Assume all steps before $n+1$)
2. Assume $P(k) \forall k \in \{1, 2, ..., n\}$ then try to show P(n+1)

Proposition 3. *A chocolate bar with $n \geq 1$ pieces can be broken into individual pieces by making n-1 breaks.*

Proof. Using weak induction,

1. base case: n = 1
   1 piece can be broken into individual pieces by making 0 breaks.
2. inductive step
   Assume P(n)="a bar with n pieces can be broken into individual pieces by making n-1 breaks"
   WTS: P(n+1)="a bar with n+1 pieces can be broken into individual pieces by making n breaks"
   The issue is that we need to know that everything from P(n) to P(1) works.
   Since we cannot prove this for an arbitrary n, we must use strong induction.

□

Proof. Using strong induction,

1. base case: n = 1
   1 piece can be broken into individual pieces by making 0 breaks.
2. inductive step
   Assume P(k) $\forall k \in \{1, 2, ..., n\}$
   WTS: P(n+1)="a bar with n+1 pieces can be broken into individual pieces by making n breaks"

   (a) Consider an arbitrary bar of n+1 size. Break the bar into two pieces
      i. One piece has k pieces
      ii. The other piece has $(n+1) - k$ pieces
   (b) Assuming $P(k)$, the first piece can be broken into individual pieces by making $k-1$ breaks.
   (c) $P(n+1-k)$ will require $n+1-k-1 = n-k$ breaks.

   $\therefore$ The total number of breaks is $1 + (k-1) + (n-k) = n$.

□

Theorem 2. *It is true that strong induction $\rightarrow$ weak induction*

Theorem 3. *Fundamental Theorem of Arithmetic:*

$$\forall n \in \mathbb{N} - 0, 1, n \textit{ is either prime or can be written as a product of primes.}$$

Proof. Using strong induction,

1. base case: n = 2
   2 is prime.
2. inductive step: Assume P(k) $\forall k \in \{2, 3, ..., n\}$
   WTS: P(n+1)
   We can prove this by cases:
   (a) n+1 is prime: It can be expressed as $1 \times (n + 1)$
   (b) n+1 is not prime

$$\exists l, w \in \mathbb{Z}, n + 1 = lw$$
$$1 < l, w < n + 1$$

So P(l) = T, P(w) = T, therefore:
$l = p_1^{k_1} \times p_2^{k_2} \times ... \times p_n^{k_n}$, where $p_1, p_2, ..., p_n$ are primes and $k \in \mathbb{N}$.
$w = q_1^{j_1} \times q_2^{j_2} \times ... \times q_m^{j_m}$, where $q_1, q_2, ..., q_m$ are primes and $j \in \mathbb{N}$.

Given l and w, we can find $n + 1$ by multiplying them together.

$$n + 1 = lw$$
$$n + 1 = (p_1^{k_1} \times p_2^{k_2} \times ... \times p_n^{k_n})(q_1^{j_1} \times q_2^{j_2} \times ... \times q_m^{j_m})$$

$n + 1$ is a product of primes.

□

Proposition 4. *Consider the sequence* $a_1 = 0, a_2 = 1, a_n = 2a_{n-1} - a_{n-2}$. *Prove* $a_n = n - 1$.

Proof. Using strong induction,

1. bsae case: n = 1, n=2
   $n = 1 : a_1 = 0 = 1 - 1$

□