Discrete Math: Homework 7

Due on November 2, 2023 at 11:59pm

Tuesday/Thursday 11:00-12:15, Phillips 383

Reese Lance - Section 003

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Unit 2.3

#44 Let f be the function from $\mathbb{R}$ to $\mathbb{R}$ defined by $f(x) = x^2$. Find

a) $f^{-1}(\{1\})$

b) $f^{-1}(\{x \mid 0 < x < 1\})$

c) $f^{-1}(\{x \mid x > 4\})$

Note: $f^{-1}(x) = \{\pm\sqrt{x}\}$

a) {1, -1}

b) $\{x \neq 0 \mid -1 < x < 1\}$

c) $\{x \mid (x > 2) \vee (x < -2)\}$

#46 Let f be a function from A to B. Let S and T be subsets of B. Show that

a) $f^{-1}(S \cup T) = f^{-1}(S) \cup f^{-1}(T)$

Proof. We will use the definition of $f^{-1}(S)$ to prove this.

- $\rightarrow$ Let $x \in f^{-1}(S \cup T)$. Then $f(x) \in S \cup T$. Thus $f(x) \in S$ or $f(x) \in T$. Thus $x \in f^{-1}(S)$ or $x \in f^{-1}(T)$. Thus $x \in f^{-1}(S) \cup f^{-1}(T)$.
- $\leftarrow$ Let $x \in f^{-1}(S) \cup f^{-1}(T)$. Then $x \in f^{-1}(S)$ or $x \in f^{-1}(T)$. Thus $f(x) \in S$ or $f(x) \in T$. Thus $f(x) \in S \cup T$. Thus $x \in f^{-1}(S \cup T)$.

Therefore, $f^{-1}(S \cup T) = f^{-1}(S) \cup f^{-1}(T)$. □

b) $f^{-1}(S \cap T) = f^{-1}(S) \cap f^{-1}(T)$

Proof. We will use the same method as part a) to prove this.

- $\rightarrow$ Let $x \in f^{-1}(S \cap T)$. Then $f(x) \in S \cap T$. Thus $f(x) \in S$ and $f(x) \in T$. Thus $x \in f^{-1}(S)$ and $x \in f^{-1}(T)$. Thus $x \in f^{-1}(S) \cap f^{-1}(T)$.
- $\leftarrow$ Let $x \in f^{-1}(S) \cap f^{-1}(T)$. Then $x \in f^{-1}(S)$ and $x \in f^{-1}(T)$. Thus $f(x) \in S$ and $f(x) \in T$. Thus $f(x) \in S \cap T$. Thus $x \in f^{-1}(S \cap T)$.

Therefore, $f^{-1}(S \cap T) = f^{-1}(S) \cap f^{-1}(T)$. □

Unit 4.1

#12 Prove that if a is a positive integer, then 4 does not divide $a^2 + 2$.

Proof. We will conduct a proof by contradicition. Assume $4 \mid a^2 + 2$. This means that $\exists k \in \mathbb{Z}, 4k = a^2 + 2$. We can rewrite this as $a^2 = 4k - 2 = 2(2k - 1)$. This means that a^2 is even which means that a is even. This means that $\exists j \in \mathbb{Z}, a = 2j$. We can rewrite this as $a^2 = 4j^2$. This means that a^2 is divisible by 4. This means that $4 \mid a^2$. This means that $4 \mid a^2 + 2$ and $4 \mid a^2$. We can then use identites to show that $4 \mid a^2 - (a^2 + 2) \equiv 4 \mid 2$. This is a contradiction because $4 \nmid 2$. Therefore, $4 \nmid a^2 + 2$. □

#30 Find the integer a such that

a) $a \equiv 43 \pmod{23}$ and $-22 \leq a \leq 0$

$$a = -3$$

b) $a \equiv 17 \pmod{29}$ and $-14 \leq a \leq 14$

$$a = -12$$

c) $a \equiv -11 \pmod{21}$ and $90 \leq a \leq 110$

$$a = 94$$

#34 Decide whether each of these integers is congruent to 3 modulo 7.
Note: $a \equiv b \pmod{n}$ means that $n \mid (a - b)$.

a) 37

$$37 \not\equiv 3 \pmod{7} \text{ because } 7 \nmid (37 - 3) = 34$$

b) 66

$$66 \equiv 3 \pmod{7} \text{ because } 7 \mid (66 - 3) = 63$$

c) -17

$$-17 \not\equiv 3 \pmod{7} \text{ because } 7 \nmid (-17 - 3) = -20$$

d) -67

$$-67 \equiv 3 \pmod{7} \text{ because } 7 \mid (-67 - 3) = -70$$

#44 Show that if n is an integer then $n^2 \equiv 0$ or $1 \pmod 4$.

Proof. We will conduct a proof by cases on n.

n is even

$$\begin{aligned} n = 2k \text{ for some } k \in \mathbb{Z} \\ n^2 = 4k^2 = 4(k^2) \\ \exists \bar{k} \in \mathbb{Z}, k^2 = \bar{k} \\ n^2 = 4\bar{k} \\ n^2 \equiv 0 \pmod 4 \end{aligned}$$

n is odd

$$\begin{aligned} n = 2k + 1 \text{ for some } k \in \mathbb{Z} \\ n^2 = 4k^2 + 4k + 1 = 4(k^2 + k) + 1 \\ \exists \bar{k} \in \mathbb{Z}, k^2 + k = \bar{k} \\ n^2 = 4\bar{k} + 1 \\ n^2 \equiv 1 \pmod 4 \end{aligned}$$

Therefore, $n^2 \equiv 0$ or $1 \pmod 4$. □

#46 Prove that if n is an odd positive integer, then $n^2 \equiv 1 \pmod 8$.

Proof. We will conduct a direct proof. Let n be an odd positive integer.

$$\exists k \in \mathbb{Z}, n = 2k + 1$$
$$n^2 = 4k^2 + 4k + 1 = 4(k^2 + k) + 1 = 4(k)(k + 1) + 1$$

Note, either k or $k + 1$ must be even, therefore

$$\exists \bar{k} \in \mathbb{Z}, k(k + 1) = 2\bar{k}$$
$$n^2 = 4(2\bar{k}) + 1 = 8\bar{k} + 1$$
$$n^2 \equiv 1 \pmod 8$$

□